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CALCULATING THE PERFORMANCE RELIABILITY OF SYSTEMS CONTAINING A LARGE NUMBER OF ELEMENTS

by V. I. Siforov

(Moscow)

The systems currently in wide use contain a large number of elements, and the damage to one of them affects the performance of the entire system.

A typical example of this type of system is a radio relay system containing a large number of intermediate stations, each of which in turn consists of a large number of electron tubes and other parts.

To calculate the reliability of such complicated systems with given characteristics of their individual elements, such as the tubes for example, we must know the quantitative ratio between the probability of damage to the entire system and the probable damage to its individual elements.

The purpose of this article is to find such a quantitative ratio and use it as a basis for the methods of calculating the reliability of systems containing a large number of elements.

1. General ratios between the probabilities of damage to the system and to its elements. Let the block diagram of a system consist of a large number n of elements A_1, ..., A_n (Fig. 1). We shall designate the probability of their damage during the time T as p_1,..., p_n, respectively. Each of these probabilities is a function of T.

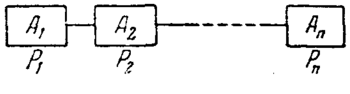

Fig. 1

[redacted]

The event in which any element A_k of the system will function normally in the period of time T is the direct opposite of the event in which that element will be damaged during the same period of time. The probability of the normal operation of element A_k during the time T will therefore be equal to $1 - p_k$.

The probability that all the n elements of the system will be operating normally during the time T will, according to the theorem of the multiplication of probabilities, be equal to

$$(1-p_1)(1-p_2)\ldots(1-p_n)$$

and the probability of damage to at least one of the elements $A_1, \ldots, A_n$ will be

$$y = 1-(1-p_1)(1-p_2)\ldots(1-p_n) \tag{1.1}$$

Since the damage to one or several elements will disrupt the normal operation of the entire system, the magnitude y represents the damage probability of the entire system.

Formula (1.1) combines the damage probability of the entire system with that of its individual elements.

In particular cases, when all the elements of the system are the same, formula (1.1) will look like this:

$$y = 1-(1-p)^n \tag{1.2}$$

or, in the case of fairly large $n (n \gg 1)$ values and not very large $p (np \lessapprox 1)$ values,

$$y = 1-\left(1-\frac{np}{n}\right)^n \approx 1-e^{-np} \tag{1.3}$$

As indicated in formula (1.1), the following conditions must be fulfilled in order to keep the probability of damage to the entire system y fairly small (less than 0.01, for example):

$$p_1 \ll 1, \quad p_2 \ll 1, \ldots, \quad p_n \ll 1$$

With these conditions fulfilled, formula (1.1) may be represented as follows:

$$y = 1 - [1 - (p_1 + p_2 + \cdots + p_n) + \cdots]$$

or

$$y \approx p_1 + p_2 + p_3 + \cdots + p_n \tag{1.4}$$

that is when the probability of damage to all the elements of the system is small, the probability of damage to the entire system during time T is approximately equal to the sum of the damage probabilities of its elements during the same period of time T.

In particular cases, when all the elements of the system are the same, formula (1.4) will look like this:

$$y \approx np \tag{1.5}$$

in which case $p \ll 1$, $np \ll 1$, and n may represent any but not necessarily large value.

Formulas (1.4) and (1.5) reveal that a high operational reliability of a system consisting of a large number of elements requires a very high level of operational reliability on the part of all its elements. For example, assuming the damage probability of the entire system to be y = 0.01 and the number of elements n = 100, the damage probability of each element will be $p = 10^{-4}$. Hence it follows that tubes designed for use in major radio relay systems must be up to very rigid standards; in any case, the probability of each individual tube getting out of commission during a given period of operation should not exceed 10^{-4}.

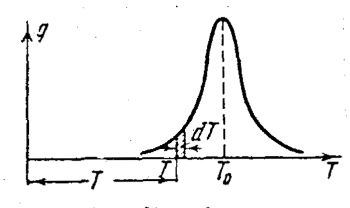

Fig. 2

2. <u>The duration of the system's operation without damage</u>. Let us find the quantitative ratio required for calculating the time of the system's operation without damage on the assumption that we know the law of probability governing the efficient operation of each element.

We shall designate as q(T) the distribution density of the probable periods of efficient operation of the system's element (Fig. 2). Then the probability that the period of uninterrupted operation of an element is found between T and T + dT will be equal to q(T)dT; that probability is graphically represented by the dotted area in Fig. 2.

The probability that the period of uninterrupted operation of an element in the system will be found somewhere between zero and T or the probability of damage to one element of the system during time T, which is the same, will be expressed by the following integral

$$p(T) = \int_0^T q(T)\,dT$$

Assuming that the law of probability governing the distribution of the periods of efficient operation of each element of the system q(T) is normal, we will have

$$q(T) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(T-T_0)^2}{2\sigma^2}\right) \tag{2.1}$$

where T is the duration of uninterrupted operation of an element in the system, T_o the mathematical expectancy of uninterrupted operation of an element in the system (the average service life of the element), and σ^2 the time variability of the efficient operation of an element (the average value of the square of the deviation of the element's service life from its average service life).

Under the normal law of probabilities, the probable damage to an element during time T will be

$$p(T) = \int_{-\infty}^{T} q(T)\,dT = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{T} \exp\left(-\frac{(T-T_0)^2}{2\sigma^2}\right) dT \qquad (2.2)$$

In the case of a small dispersion of the periods of efficient operation of every element in the system or a high degree of similarity of the elements (in point of their service life), which is the same, the probability of damage to an element in the system during time T can be approximately expressed by the following integral

$$p(T) = \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{T} \exp\left(-\frac{(T-T_0)^2}{2\sigma^2}\right) dT \qquad (2.3)$$

as in such cases

$$\int_{-\infty}^{0} \exp\left(-\frac{(T-T_0)^2}{2\sigma^2}\right) dT \ll \int_{0}^{T} \exp\left(-\frac{(T-T_0)^2}{2\sigma^2}\right) dT$$

Thus in the case of relatively small dispersions both formulas (2.2) and (2.3), provide approximately the same value of the probable damage to systems p(T). But in the case of larger dispersions, the mentioned formulas cannot be used, as in such cases the law of probabilities governing the distribution of the periods of the element's efficient operation is different from the normal law.

Let us assume that the system consists of n similar elements. From formulas (1.5) and (2.2) we will find the following probability of damage to the system during time T

$$y = \frac{n}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{1}{2}x^2}\,dx \qquad \left(\frac{T_0 - T}{\sigma} = x\right) \qquad (2.4)$$

As

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{1}{2}x^2}dx + \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{1}{2}x^2}dx = \frac{1}{2}$$

we will have

$$\frac{y}{n} = \frac{1}{2} - \Phi(x) \qquad \left(\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-\frac{1}{2}x^2}dx\right) \qquad (2.5)$$

Here $\Phi(x)$ is the distribution function known in the theory of probabilities, and x is known according to (2.4).

On the basis of (2.5) and the known tables of the $\Phi(x)$ function we shall cite, for the sake of convenience, the numerical relation between y/n and $x=(T_0-T)/\sigma$:

$x=$	2.0	2.5	3.0	3.2	3.4
$y/n=$	2.28×10^{-2}	6.2×10^{-3}	1.35×10^{-3}	6.9×10^{-4}	3.4×10^{-4}
$x=$	3.6	3.8	4.0	4.5	5.0
$y/n=$	1.6×10^{-4}	7.2×10^{-5}	3.2×10^{-5}	3×10^{-6}	3×10^{-8}

Assuming that we know the average service life of the element T_0, the average value of the square of the deviation of the element's service life from its average service life (that is, σ^2) and the n number of elements of the system, formula (2.5) will make it possible to calculate the damage probability y of the entire system during time T. Seeking to establish the admissible damage probability y of the system as a whole and known the number of elements n of the system as well as the parameters T_0 and σ characterizing each element, it is also possible to establish the time of the system's normal operation by the following formula

$$T=T_0-x\sigma \tag{2.6}$$

which follows from the relation (2.4).

We shall cite several examples to illustrate the found quantitative relations

Let the average service period of an element of the system be T_0 = 10,000 hours, and the average square of the deviation from the service period σ = 1,000 hours. The probability of damage to an element of the system during $T_0 - \sigma$ = 9,000 hours, according to (2.5), will be

$$p=\frac{y}{n}=\frac{1}{2}-\Phi(1)=0.16 \tag{2.7}$$

The probability of an element of the system having a service life from $T_0 + \sigma$ = 11,000 hours and more, in view of the symmetry of the probability distribution curve, will also be equal to 0.16. Consequently, the probability of the service life of an element of the system being between $T_o - \sigma$ and $T_o + \sigma$, that is between 9,000 and 11,000 hours, is $1 - 2 \times 0.16 = 0.68$. Assuming that the number of elements in the system is $n = 100$ and the admissible probability of its damage $y = 0.01$, we will get $y/n = 10^{-4}$. Using the above cited numerical relationship between y/n and x:

$$x = \frac{T_0 - T}{\sigma} \approx 3.7$$

and according to (2.6), the period of the normal operation of the system will be

$$T = T_0 - x\sigma = 10\,000 - 3.7 \times 1000 = 6300 \text{ hours.}$$

The utilization factor of the system's element, that is, the ratio between the period of its operation and its average service life, will be

$$\eta = \frac{6300}{10\,000} = 0.63$$

Let us assume now that $\sigma = 500$ hours while T_o, y and n retain their previous values. The period of the system's normal operation will then be

$$T = \eta T_0 = 10\,000 - 3.7 \times 500 = 8150 \text{ hours,}$$

and the utilization factor of the system's element is

$$\eta = \frac{8150}{10\,000} = 0.815$$

At very small σ values, according to (2.6), the time of the system's normal operation will be close to T_o, and the utilization factor of each element will be close to 1.

It follows from the above-cited numerical examples that a reduction of the heterogeneity factor σ/T_o of each element of the system in point of service life from 0.1 to 0 increases the utilization factor of the element from 0.63 to 1, that is relatively little. This justifies the conclusion that it is not expedient to have each element of the system meet more rigid requirements in point of similarity of service life than those corresponding to the heterogeneity factor $\sigma/T_o = 0.1$.

With $\sigma/T_o = 0.2$, that is $\sigma = 2{,}000$ hours ($T_o = 10{,}000$ hours), we will get

$$T = 10\,000 - 3.7 \times 2000 = 2600 \text{ hours}; \qquad \eta = \frac{2600}{10\,000} = 0.26$$

It can easily be seen that an increase of the heterogeneity factor σ/T_o, beginning with $\sigma/T_o = 0.2$, is accompanied by a sharp reduction of the utilization factor of the system's element. Hence the conclusion that the heterogeneity factor σ/T_o of each element of the system should not exceed 0/2 in point of service life.

To find out to what extent the period of the system's operation without damage will be reduced by assigning more rigid requirements to its damage probability, we shall assume that $y = 0.001$. Retaining the previous assumption that $n = 100$, we will get $y/n = 10^{-5}$. Making use of the above-cited numerical relationship, we will find that $x = 4.3$. The normal operation period of the system (with $T_o = 10{,}000$ hours and $\sigma = 1{,}000$ hours) is

$$T = T_0 - x\sigma = 10\,000 - 4.3 \times 1000 = 5700 \text{ hours},$$

and the utilization factor of one element of the system is

$$\eta = \frac{5700}{10\,000} = 0.57$$

Comparing these results with the previous ones, we can see that a reduction of the damage probability of the system from 0.01 to 0.001, that is a tenfold reduction, reduces the utilization factor of the system's element from 0.63 to 0.57, which is quite insignificant. Hence it would be practical to assign very rigid requirements to the damage probability of the system, as this would insure a high degree of its operational reliability and a relatively little deterioration of the utilization of its elements (the tubes, for example).

3. <u>An analysis of various methods of replacing the system's elements.</u> The numerical ratios found in the previous section apply to the case when the elements, such as tubes, are replaced simultaneously in the entire system. By this method, all the elements of the system are replaced at intervals of $T = \eta T_o$, where T_o represents the average service life of an element, and η its utilization factor.

There are also other methods of replacing the elements. One of them is to replace them in groups. By this method all the elements of the system are divided into k groups. The elements within each group are replaced simultaneously but each group is replaced at a different time. For example, if all the elements of the system were divided into two identical groups, their replacement in the first group would take place at t = 0, t = T, t = 2T, etc., and in the second group at t = ½T, t = 3/2T, t = 5/2T, etc.

The question naturally arises as to which element-replacement method is better, the one that calls for a division into groups or the other? In other words, which of the mentioned methods will provide for a fuller utilization of the system's elements (especially the tubes)? To answer these questions, we must find the damage probability of the entire system under the group-replacement method.

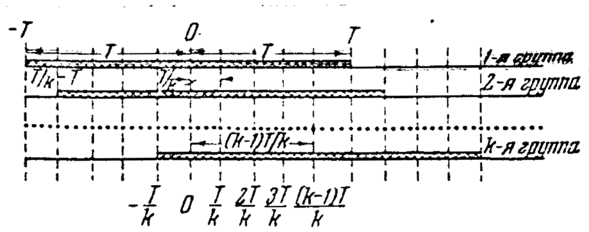

Fig. 3 [группа = group]

The diagram in Fig. 3 illustrates the replacement of the elements of the system by dividing it into k number of similar groups each of which contains n/k elements. The elements in any group are replaced at intervals of $T = \eta T_o$, but the replacement time of the elements in the neighboring groups (by their number) is shifted by T/k. In the first group the elements are replaced at $t = 0$, in the second at $t = T/k$, in the third at $t = 2T/k$ and in the k group at $t = (k - 1)T/k$.

The probability y_1 of damage to the first group in the period of time between $t = 0$ and $t = T$, according to (1.2), will be

$$y_1 = 1 - [1 - p(T)]^{n/k}$$

where p(T) is the probability of damage to one element during time T.

The probability z_1 of normal operation of the first group in the period between $t = 0$ and $t = T$ will be

$$z_1 = 1 - y_1 = [1 - p(T)]^{n/k} \qquad (3.1)$$

The following two conditions must be observed if the second group is to work normally in the period between $t = 0$ and $t = T$: 1) the normal operation of this group in the period between $t = 0$ and $t = T/k$; 2) its normal work in the period between $t = T/k$ and $t = T$. The observance of the first condition, in turn, requires the normal operation of the second group from the time of the previous replacement of its elements, that is from $t = (T/k) - T$ to $t = 0$, and from $t = 0$ to $t = T/k$. In other words, the observance of the first condition calls for the normal operation of the second group in the period of time between $(T/k) - T$ and T/k of duration T. The probability z_2' of observing the first condition will be

$$z_2' = [1 - p(T)]^{n/k} \qquad (3.2)$$

and the probability of observing the second condition

$$z_2'' = \left\{1 - p\left[\frac{(k-1)T}{k}\right]\right\}^{n/k} \qquad (3.3)$$

$$p\left[\frac{(k-1)T}{k}\right]$$

is the probability of damage to one of the elements during $T - (T/k)$.

According to the multiplication theory, the probability z_2 of normal operation of the second group in the period between t = 0 and t = T will be

$$z_2 = z_2' z_2'' = [1 - p(T)]^{n/k} \left\{1 - p\left[\frac{(k-1)T}{k}\right]\right\}^{n/k} \tag{3.4}$$

Similar to the probability z_3, z_4,..., z_k of the normal operation of the 3rd, 4th,... k groups, respectively, in the period of time between t = 0 and t = T will be

$$z_3 = [1 - p(T)]^{n/k} \left\{1 - p\left[\frac{(k-2)T}{k}\right]\right\}^{n/k} \tag{3.5}$$

$$z_4 = [1 - p(T)]^{n/k} \left\{1 - p\left[\frac{(k-3)T}{k}\right]\right\}^{n/k}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$z_k = [1 - p(T)]^{n/k} \left\{1 - p\left(\frac{T}{k}\right)\right\}^{n/k}$$

According to the multiplication theory, the probability z of the normal operation of all the k groups, that is of the entire system, between t = 0 and t = T will be z = z_1, z_2..., z_k, or, by substituting here the expression for z_1, z_2,... z_k, we will find

$$z = [1 - p(T)]^n \prod_{i=1}^{k-1} \left\{1 - p\left[\frac{(k-i)T}{k}\right]\right\}^{n/k} \tag{3.6}$$

The damage probability y of the entire system in the period between t = 0 and t = T will be

$$y = 1 - z = 1 - [1 - p(T)]^n \prod_{i=1}^{k-1} \left\{1 - p\left[\frac{(k-i)T}{k}\right]\right\}^{n/k} \tag{3.7}$$

When the damage probability p of each element of the system in various periods of time is known, this formula makes it possible to determine the damage probability y of the entire system during time T occasioned by the replacements of its elements k in similar groups.

Expanding the right part of formula (3.7) into a series by p degrees, discarding the low magnitudes of high orders and considering the probabilities p as low magnitudes of a high order, we will get

$$y = np(T) + \frac{n}{k} \sum_{i=1}^{k-1} p\left[\frac{(k-i)T}{k}\right] \tag{3.8}$$

Comparing this formula with (1.5), we can see that, all other conditions being equal, the probability of damage to the system when its elements are replaced in groups is greater than the probability of damage to the same system when its elements are replaced without being divided into groups.

As probability p is an increasing function of its argument,

$$p\left[\frac{(k-i)T}{k}\right] < p(T) \tag{3.9}$$

at any i value from i = 1 to i = k - 1.

Taking this inequality into consideration, the relationship (3.8) can be rewritten as follows

$$y < np(T) + \frac{n}{k}(k-1)\,p(T)$$

or, since $\frac{n}{k}(k-1)\,p(T) < np(T)$, as follows

$$y < 2np(T) \tag{3.10}$$

Comparing this formula with (1.5), we may conclude that a change from the simultaneous replacement of the elements in the entire system to changing them by groups increases the damage probability of the system by less than 100% regardless of the number of groups.

It should be pointed out that this conclusion is valid only for the probability of damage in the period of time between 0 and T. Generally speaking, the probabilities of damage during the same period of time depend on the initial and final moments of that period. But the difference in the probabilities connected with the simultaneous replacement of the elements and their replacement by groups at any initial moments of time is of little practical significance. Bearing in mind the practical convenience of the group method, it may be assumed that its utilization would be expedient in many cases.

In the above-discussed element-replacement methods, the duration T of the work of each element constitutes a certain part of its average service life T_0, that is each element is used only partially. Let us see to what extent the probability of damage to the system would increase if each of its elements were fully utilized. In this case, after a fairly long period of time following the simultaneous replacement of all the elements in the system and after the replacement of a large number of damaged elements at different periods of time, any element of the system may be damaged at any moment of time (between t = 0 and t = T) with equal probability. In other words, if the moment of damage to any element of the system were designated by a point on segment OT of time axis which is T long, that point might fall in any place of the mentioned segment with equal probability.

The mathematical expectation λ of the number of damaged elements during time T will be

$$\lambda = n\frac{T}{T_0} \tag{3.11}$$

where n represents the total number of elements in the system, and T_0 the mathematical expectation of the length of service of each element.

In case $\lambda \ll 1$, the probability of damage to the system during time T, that is the probability of at least one element of the system being damaged during time T, will be approximately equal to the mathematical expectation λ. We can therefore write

$$y = n\frac{T}{T_0} \quad \text{or} \quad T = \frac{yT_0}{n} \tag{3.12}$$

Knowing the number of elements n and the average service life T_0 of each element, and seeking to establish the admissible probability value y of the damage to the system, this relationship enables us to compute the duration of the system's normal operation. Assuming that $n = 100$, $y = 0.01$ and $T_0 = 10,000$ hours, we will find that $T = 1$ hour. It was shown earlier (see 2) that with the same probability of damage to the system, the same number of elements and the same average service life of each element, duration T of the system's operation amounted to 6,300 hours. From this it follows that, other conditions being equal, the change from a partial to a full utilization of each element reduces the duration of the system's operation to a very considerable extent. In the above-discussed example, the change of the utilization factor η of each element from 0.63 to 1 reduces the duration of the normal operation of the system with a preselected damage probability of 0.01 6,300 times.

It is clear from the above discussion that in systems containing a large number of elements, the full utilization of each element (in point of service life) is impractical as it will reduce the duration of the system's normal operation to a very considerable extent. This conclusion is, of course, valid only in cases when the damage to at least one element of the system disrupts its normal operation.

4. <u>Some methods of designing systems with an automatic replacement of the elements</u>. One of the methods facilitating the full utilization of each element of the system, as well as a small damage probability over a long period of time, is the method of the automatic replacement of the damaged elements.

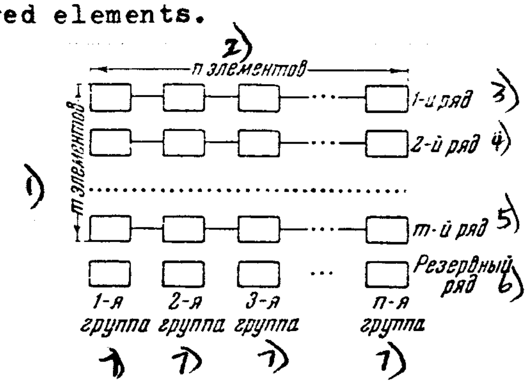

Fig. 4

Legend:
1) m elements
2) n elements
3) 1st series
4) 2nd series
5) m series
6) reserve series
7) group

Fig. 4 shows a block-diagram of a system with automatic stand-by provisions consisting of n groups with m elements in each group. Each group also has one reserve element. If any element in a particular group is damaged, it is automatically disconnected and the reserve element of the same group is connected in its place. The time required for the automatic replacement of the damaged element by the reserve element can be very short. Such an automatic replacement practically does not affect the operation of the system. The next step is to repair the damaged element (the replacement of tubes, for example) and connect it in the system in place of the reserve element which, in turn, is automatically connected again in case of damage to any other element of this group.

Designating by T_p the time between the damage of any element of the system and the reconnection of the repaired element, we can write the following formula in accordance with (3.12)

$$y_p = m\frac{T_p}{T_0} \qquad T_p = \frac{y_p T_0}{m} \tag{4.1}$$

where Y_p is the probability that at least one of the m elements of this group will go out of commission during the repair period T_p, that is the probability of the normal operation of the system being disrupted.

Knowing the m number of elements in the group, the average service life of each element, seeking to establish the admissible probability Y_p of the normal operation of the system being disrupted and making use of formula (4.1), it is possible to calculate the admissible duration T_p of repair to the damaged element. For example, assuming that $m = 5$, T_o = 10,000 hours and $Y_p = 0.01$, we will get $T_p = 20$ hours.

Submitted on 17 May 1954.